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Supersymmetric distributions, Hilbert spaces of supersymmetric functions and quantum fields

Florin Constantinescu

Fachbereich Mathematik, Johann Wolfgang Goethe-Universität Frankfurt,
Robert-Mayer-Strasse 10, D 60054 Frankfurt am Main, Germany

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Abstract

The recently investigated Hilbert–Krein and other positivity structures of the superspace are considered in the framework of superdistributions. These tools are applied to problems raised by rigorous supersymmetric quantum field theory.

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1. Introduction

In spite of their formidable success in many areas of physics, path integral methods are in most cases formal. In the case of supersymmetry, even methods of moderate rigour like canonical quantization are not generally worked out. Certainly supersymmetric quantum fields have to be operators but the (positive definite) Hilbert space, in which they are supposed to act, and the associated domain problems are outside formal methods. Whereas it is easy to see that the Hamiltonian in supersymmetry is positive [1, 2], the argument does not give further information on the nature and realization of the Hilbert space behind.

In [3] we have proved that the $N = 1$ superspace shows an intrinsic Hilbert–Krein structure realized on supersymmetric functions (not fields). Taking up this finding, in this note, we answer general questions on supersymmetric distributions and Hilbert spaces together with their first applications to what was called axiomatic quantum field theory [4]. They have to get definite answers before pursuing further developments.

In this paper we concentrate on the ‘massive’ case in quantum field theory. The ‘massless’ case requires a little more effort but does not require new tools. The main point of the massless case is a natural restriction of the space of supersymmetric functions to a subspace of it [3]. This restriction is necessary in order to maintain the intrinsic Hilbert–Krein structure and on the other hand reminds rigorous methods of the canonical quantization of gauge theories (Gupta–Bleuler, Nielsen–Lautrup, Kugo–Ojima etc, see [3]).

Before starting work we discuss in some details the content of the present paper pointing out its necessity and usefulness and making reference to alternative constructions.

Supersymmetry is a large field of activity in physics mostly related to quantum field and string theory. With few exceptions related to representation theory of super-algebras and groups and its consequences, supersymmetry is investigated by path integral methods. In supersymmetry these powerful methods include integration over the non-commutative (Grassmann) variables being even more formal than the same methods in the non-supersymmetric case. On the other hand in the non-supersymmetric case there are alternative methods which preceded path integrals and in the modern approach can be looked at as complementary to path integral methods. Canonical quantization, the LSZ-formalism and the axiomatic approach are, among others, well-known examples. Such methods are convenient and even necessary if we want to pass to the S-matrix of the theory under study. In particular proving unitarity of the S-matrix is a top achievement in quantum field theory which, at least from a rigorous point of view, which seems to be outside formal path integrals methods. Unitarity requires a Hilbert space framework induced by a (physical) positive definite scalar product. In the non-supersymmetric case it is a general belief supported by C^* -methods in quantum field theory that the Hilbert space can be realized on functions of spacetime (wavefunctions). The most natural question to ask is if this holds in the supersymmetric case too, i.e. to what extent there are positive definite (invariant) scalar products on supersymmetric functions where in the definition of the scalar product the integration is performed not only on spacetime but also on Grassmann variables (Berezin integration). The reader can easily experience that this requirement is not easy to achieve. It was proved recently [3] that at least in the case $N = 1$ the superspace comes equipped with a natural (invariant) indefinite Krein structure which generates as usual (see for instance [5]) a positive definite Hilbert space structure.

In the present paper we propose ourself to exploit the Hilbert–Krein structure of the $N = 1$ supersymmetry starting a study of supersymmetric quantum field theory outside path integral methods. There are two types of problems which we address in this paper. The first type is concerned with consistency problems related to the definition of quantum fields as operator-valued superdistributions. In particular in section 3 difficulties with tensor products of superdistributions and nuclearity are shown to be absent. Furthermore it is shown that C^* -methods are applicable to supersymmetry, an interesting point being the Hermiticity of the scalar product generated by the Gelfand–Neumark construction in section 4. The second type of problems studied in this paper concerns information on supersymmetric Hilbert space operators especially those which appear in quantum field theory. In particular in section 8 we study the simplest example of a free field whereas in section 9 the general form of the interacting supersymmetric n -point functions is worked out. For $n = 2$ it gives the supersymmetric Källén–Lehmann representation. Some of the aspects of quantum superfields appear in section 10.

Concluding we may say that there are two points which single out the results of the present paper: they are new results on free and interacting supersymmetric quantum fields obtained outside path integral methods and they are rigorous.

2. Test functions and distributions for supersymmetry

Our supersymmetric (test) functions and distributions and the Hilbert spaces related to them are not as general as usually used in the mathematical literature on supersymmetry and supermanifolds [6–8]. In our work we are guided by direct applications to the supersymmetric quantum field theory [1]. Let x be in the four-dimensional Minkowski space with the signature $(-1, 1, 1, 1)$ and let $\theta, \bar{\theta}$ be two-component Grassmann variables associated with x . We use the notations and conventions in [1]. They coincide with those of [9] up to the Pauli σ^0 which in [1] and this paper is equal to the 2×2 -matrix minus one (it is one in [9]). It means that

in this paper $\bar{\sigma}$ is defined to be $\bar{\sigma} = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3) = (-1, -\sigma^1, -\sigma^2, -\sigma^3)$ where $\sigma = (\sigma^l), l = 0, 1, 2, 3$ are the Pauli matrices. The most general function of $z = (x, \theta, \bar{\theta})$ is of the form

$$\begin{aligned} X(z) &= X(x, \theta, \bar{\theta}) \\ &= f(x) + \theta\varphi(x) + \bar{\theta}\bar{\chi}(x) + \theta^2 m(x) + \bar{\theta}^2 n(x) \\ &\quad + \theta\sigma^l\bar{\theta}v_l(x) + \theta^2\bar{\theta}\bar{\lambda}(x) + \bar{\theta}^2\theta\psi(x) + \theta^2\bar{\theta}^2 d(x) \end{aligned} \quad (2.1)$$

where the coefficients of $\theta, \bar{\theta}$ are functions of x . For the vector component v we can write equivalently

$$\theta\sigma^l\bar{\theta}v_l = \theta^\alpha\bar{\theta}^{\dot{\alpha}}v_{\alpha\dot{\alpha}}$$

where

$$v_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^l v_l, v^l = -\frac{1}{2}\bar{\sigma}^{l\dot{\alpha}\alpha} v_{\alpha\dot{\alpha}}.$$

In principle x (or more precisely each of its four components) is a number but at a certain stage of the computations related to supersymmetry we will be forced to admit that x is not only a number (base) but also contains an even element of the Grassmann algebra. In this case we perform the Taylor expansion retaining for x only the base. Functions of several variables z_1, z_2, \dots have also expansions in $\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2, \dots$ of the same type. In what follows we use a mixed van der Waerden calculus in which $\theta, \bar{\theta}$ are Grassmann variables commuting with the component of the ‘spinor’ coefficients of X . The rules of this calculus are [3]:

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = -\chi^\alpha\psi_\alpha = -\chi\psi \quad (2.2)$$

$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = -\bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = -\bar{\chi}\bar{\psi} \quad (2.3)$$

$$\overline{\chi\psi} = \overline{\chi^\alpha\psi_\alpha} = \bar{\chi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi} = -\bar{\chi}\bar{\psi} = -\overline{\psi\chi} \quad (2.4)$$

together with

$$(\theta\phi)(\theta\psi) = \frac{1}{2}(\phi\psi)\theta^2 \quad (2.5)$$

$$(\bar{\theta}\bar{\phi})(\bar{\theta}\bar{\psi}) = \frac{1}{2}(\bar{\phi}\bar{\psi})\bar{\theta}^2 \quad (2.6)$$

$$\chi\sigma^n\bar{\psi} = \bar{\psi}\bar{\sigma}^n\chi \quad (2.7)$$

$$\overline{\chi\sigma^n\bar{\psi}} = -\bar{\chi}\bar{\sigma}^n\psi = -\psi\sigma^n\bar{\chi} \quad (2.8)$$

$$\overline{\bar{\chi}\bar{\sigma}^n\psi} = -\chi\sigma^n\bar{\psi} = -\bar{\psi}\bar{\sigma}^n\chi. \quad (2.9)$$

They differ from the usual ones in which the Grassmann variables together with the components of the spinor fields are all anticommuting [1, 9]. In this latter case, following the terminology in physics, X is said to be a field (but not yet the operator quantum field to be defined later). As a consequence we make difference between (supersymmetric) functions (or distributions), fields and quantum fields to be defined later. Note that some combined relations as for instance

$$\overline{\chi\sigma^n\bar{\psi}} = -\bar{\chi}\bar{\sigma}^n\psi$$

remain unchanged with respect to the mixed or usual van der Waerden calculus. Taking $\bar{\theta}$ to be the (Grassmann) conjugate of $\theta, \bar{\theta}_{\dot{\alpha}} = \overline{\theta_\alpha}$, with $\overline{\theta_{1\alpha}\theta_{2\beta}\dots\theta_{n\gamma}} = \bar{\theta}_{n\dot{\gamma}}\dots\bar{\theta}_{2\dot{\beta}}\bar{\theta}_{1\dot{\alpha}}$, we have for functions

$$\begin{aligned} \bar{X} = \bar{X}(x, \theta, \bar{\theta}) &= \bar{f}(x) - \theta\bar{\chi}(x) - \bar{\theta}\bar{\varphi}(x) + \theta^2\bar{n}(x) + \bar{\theta}^2\bar{m}(x) \\ &\quad + \theta\sigma^l\bar{\theta}\bar{v}_l(x) - \theta^2\bar{\theta}\bar{\psi}(x) - \bar{\theta}^2\theta\bar{\lambda}(x) + \theta^2\bar{\theta}^2\bar{d}(x) \end{aligned} \quad (2.10)$$

where the bar represents either the complex, the Grassmann or both complex and Grassmann conjugations (the one or another meaning of the bar will be clear from the context). This relation is different from the usual one in which Grassmann variables anticommute with spinor components (i.e. the case of fields) which is

$$\begin{aligned}\bar{X} &= \bar{X}(x, \theta, \bar{\theta}) \\ &= \bar{f}(x) + \theta \chi(x) + \bar{\theta} \bar{\varphi}(x) + \theta^2 \bar{n}(x) + \bar{\theta}^2 \bar{m}(x) \\ &\quad + \theta \sigma^I \bar{\theta} \bar{v}_I(x) + \theta^2 \bar{\theta} \bar{\psi}(x) + \bar{\theta}^2 \theta \lambda(x) + \theta^2 \bar{\theta}^2 \bar{d}(x)\end{aligned}\quad (2.11)$$

by minus signs of the odd terms. The relation $\overline{\bar{X}} = X$ holds in both cases.

Now we pass to superdistributions. The cheapest way, which we adopt in the present paper, is to define them again by (2.1) where the coefficients of $\theta, \bar{\theta}$ are now distributions instead of functions. The definition of functions and distributions can be extended to several variables. The standard order of the Grassmann variables is supposed to be $\theta_1, \bar{\theta}_1; \theta_2, \bar{\theta}_2; \dots$. For convenience we denote by $S(R^{4n} \times G)$ the set of functions in n variables z_1, z_2, \dots, z_n with coefficients in $S(R^{4n})$ where G stays for Grassmann. Some problems appear at the point when we want to pass to the duality functional i.e. if we want to look at superdistributions as linear continuous functionals on spaces of supersymmetric test functions (supersymmetric duality). Certainly the duality functional has to extend integration including Grassmann variables (Berezin integration). In one variable this means

$$(\mathcal{X}, X) = \mathcal{X}(X) = \int d^8 z \mathcal{X}(z) X(z) \quad (2.12)$$

where X is the test function and \mathcal{X} would be distribution. The notations and conventions regarding the Berezin integration are as in [9]. Note that we have fixed the order in which \mathcal{X} and X appear in integral (2.12). One way to cope with duality would be to introduce a locally convex topology on the space of test functions. This can be done by using seminorms (norms) suggested in [6, 8]. Another way is the following. The superfunction X can be identified with the vector-valued function X with components $(f, \varphi, \bar{\chi}, m, n, v, \bar{\lambda}, \psi, d)$. We assume the coefficient functions to be indefinitely differentiable and may give to the linear space of vector-valued functions X the locally convex topology common for the Schwartz spaces \mathcal{D} (or S). By usual vector-valued duality we construct vector-valued distributions (linear continuous functionals) which will be denoted again by \mathcal{X} . The last construction has nothing to do with Grassmann integrals but \mathcal{X} constructed by the above-mentioned supersymmetric duality and \mathcal{X} constructed with the help of the vector-valued duality are certainly related. The relation is

$$\mathcal{X}_V(X_{\text{inv}}) = \mathcal{X}_G(X) \quad (2.13)$$

where the index V stays for vector valued, G for the result of the Grassmann integration in the duality functional (2.12) and the subscript 'inv' indicates an involution of the coefficients of X . It exchanges f with d , then φ, χ with λ, ψ and finally keeps v unchanged. It is a direct consequence of the Grassmann integration in the rhs of (2.13) which selects coefficients of the highest power $\theta^2 \bar{\theta}^2$. We do not further insist on this relation because as we will see it is only of limited help for us.

Concluding there are at least three possibilities to look at superdistributions. We argued that these possibilities are in principle related. Our discussion is sufficient in order to point out some difficulties which appear in the general framework of superdistributions. These difficulties are related to complex (and Grassmann) conjugation and to tensor products. Defining complex conjugation of usual distributions and related it to a Hilbert space structure (as this is necessary for instance in the discussion of the Gelfand triple) poses no problems. In contrast, complex conjugation of superdistributions supplemented by the Grassmann one in the

duality interpretation may pose problems because Grassmann conjugation reverses the order of the Grassmann variables. We will avoid such problems from the beginning considering only superdistributions which are even polynomials in the Grassmann variables.

3. Nuclearity in supersymmetric quantum field theory

Our interest is devoted now to the second difficulty, i.e. to the definition of the supersymmetric tensor products. Consider the product $\mathcal{X}_1(X_1)\mathcal{X}_2(X_2)$ where X_1, X_2 are superfunctions in different variables and $\mathcal{X}_1, \mathcal{X}_2$ superdistributions of type (2.1). Then for $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$ we generally have

$$\mathcal{X}(X_1X_2) \neq \mathcal{X}_1(X_1)\mathcal{X}_2(X_2). \quad (3.1)$$

This happens because X_1 generally does not commute with \mathcal{X}_2 (remember that our ‘spinorial’ coefficients being functions or distributions commute with Grassmann variables). In this case some problems with the distribution theoretic tensor products may appear. Such problems are not present if for instance \mathcal{X}_2 is an even polynomial in the Grassmann variables. In this case $\mathcal{X}(X_1X_2) = \mathcal{X}_1(X_1)\mathcal{X}_2(X_2)$ as it should be. Topological aspects of tensor products can be inferred from the vector-valued interpretation or by introducing from the beginning as mentioned above seminorms (norms) directly on functions of type (2.1) with indefinitely differentiable coefficients. Nuclearity (i.e. the kernel theorem well known in distribution theory) seems to be assured in the vector-valued interpretation but it is not what we need. Let us discuss this property in the physical framework of supersymmetric quantum field theory. Nuclearity is needed in quantum field theory mainly by introducing the n -point functions as distributions in their joint variables. Indeed if V is the quantum field operator considered as operator-valued distribution [4], Ω the vacuum and X_1, X_2, \dots, X_n test functions in $S(R^4)$, then $(\Omega, V(X_1)V(X_2) \cdots V(X_n)\Omega)$ is linear and separately continuous in X_1, X_2, \dots, X_n . By nuclearity it defines the distribution $W_n(x_1, x_2, \dots, x_n) \in S'(R^{4n})$ in $4n$ -variables x_1, x_2, \dots, x_n such that we have $(\Omega, V(X_1)V(X_2) \cdots V(X_n)\Omega) = W_n(X_1, X_2, \dots, X_n)$ where the relation between $W_n(x_1, x_2, \dots, x_n) \in S'(R^{4n})$ and $W_n(X_1, X_2, \dots, X_n)$ is the standard one. In the supersymmetric quantum field theory some problems of similar nature as in the previous section may appear because of non-commutativity of $V(z_i)$ and $X_j(z_j)$. In order to avoid them we may assume at a certain stage that $(\Omega, V(X_1)V(X_2) \cdots V(X_n)\Omega)$ can be written as multilinear form $W_n(X_1, X_2, \dots, X_n)$ separately continuous in the superfunctions X_1, X_2, \dots, X_n . This is the case if the n -point functions are even in the Grassmann variables. Then by nuclearity $W_n = W_n(z_1, z_2, \dots, z_n) \in S'(R^{4n} \times G)$. Certainly we can avoid discussing nuclearity at all by assuming that the n point functions $W_n = W_n(z_1, z_2, \dots, z_n) \in S'(R^{4n} \times G)$ with

$$(\Omega, V(X_1)V(X_2) \cdots V(X_n)\Omega) = W_n(X_1, X_2, \dots, X_n) \quad (3.2)$$

exist as superdistributions. This assumption is very natural because it reflects and captures formal computations in physics. For example computing the two-point function of the free Wess–Zumino model (directly or by means of functional integrals) we use fields with non-commuting spinor coefficients. This is equivalent in this particular case with one form or another of the above-mentioned assumption. Moreover the two-point functions turn out to be even in Grassmann variables. The n -point functions of the free field are products of (even) two-point functions and pose no problems either. The interacting case is discussed in section 9.

At this point, as a further technical remark, it is worthwhile anticipating some points of our study remembering first that elementary nuclearity is related to the idea of the Gelfand

rigged space (Gelfand triple). The most well-known Gelfand triple is $S \subset L^2 \subset S'$ where S' is the space of tempered distributions over the test function space S of rapidly decreasing functions. In supersymmetry we generally do not have this situation because (as we will see) the SUSY L^2 space off-shell does not exist. On the other hand we will show that the supersymmetric (invariant) L^2 space does exist on-shell being segregated by a Krein structure of the off-shell superspace. A key point will be the fact that general distribution theoretic tools off-shell, as discussed above, will be compatible with the Hilbert space (including Gelfand triple) considerations on-shell.

We conclude that possible difficulties with superdistributions related to complex conjugation and especially to tensor products and nuclearity do not show up in supersymmetric quantum field theory. This encourages us to look for further structures in the following sections.

4. Supersymmetric *-algebra

For the time being we stay off-shell and introduce the counterpart of a topological *-algebra used in the reconstruction theorem [4]. This is a locally convex *-algebra constructed with the help of supersymmetric functions. Suppose that the coefficients of the supersymmetric functions $X(z_1, z_2, \dots, z_n)$ belong to the Schwartz space $S(R^{4n})$. Let $\mathcal{B}(S)$ be the set of sequences (X^n) , $n = 0, 1, 2, \dots$ such that X^0 is a complex number and $X^n = 0$ for all but finitely many n 's. In the vector space $\mathcal{B}(S)$ we define the operations of multiplication and the *-operation:

$$(XY)^n(z_1, z_2, \dots, z_n) = \sum_{j=0}^n X^j(z_1, \dots, z_j) Y^{n-j}(z_{j+1}, \dots, z_n) \quad (4.1)$$

$$(X^*)^n(z_1, \dots, z_n) = \overline{X^n(z_n, \dots, z_1)} \quad (4.2)$$

where bar means at the same time complex and Grassmann conjugation. We prove that $*$ is an involution. This operation makes $\mathcal{B}(S)$ into an algebra with involution and identity. It is not a Banach algebra but it can be turned as usual into a locally convex algebra. Positivity and positive definite functionals (states) on this algebra will play a central role. We come back to the involution properties. It is clear that $*$ is conjugate antilinear. It remains to prove that it reverses the order of products $(XY)^* = Y^*X^*$ and that $(X^*)^* = X$. This follows from the relation

$$\overline{\overline{X^k(z_1, \dots, z_k)} Y^l(w_1, \dots, w_l)} = \overline{Y^l(w_1, \dots, w_l)} \overline{X^k(z_1, \dots, z_k)} \quad (4.3)$$

for $k, l = 0, 1, 2, \dots$. This algebra will be the framework of the reconstruction theorem which provides us with supersymmetric operator-valued fields. Note that for usual functions i.e. functions which do not depend on Grassmann variables we have besides (4.3) also $\overline{X^k Y^l} = \overline{X^k} \overline{Y^l}$, a relation which is generally invalid if Grassmann variables are present and the bar includes Grassmann conjugation. But this relation is not used in the Gelfand–Neumark-like construction of the reconstruction theorem which we will perform in this paper. The crucial Hermiticity of the scalar product to be constructed rests only on the relation $(XY)^* = Y^*X^*$ which holds in both usual and supersymmetric framework. This shows that supersymmetry is compatible with a quantum field theory C^* -approach and makes use of the *-algebra structure in an even more intricate way. Indeed the involution $*$ in supersymmetry includes Grassmann conjugation besides the complex one.

The reconstruction theorem itself in the supersymmetric framework does not seem to raise any special problems as this was already noted in [11]. But at this stage the reader

may feel uneasy: the positivity needed in the reconstruction theorem can be easily formulated using the $*$ -algebra above but it is not clear to what extent such a property (i.e. positivity) can be realized in a framework in which Grassmann variables and Berezin integration play a central role. Experience with Berezin integration shows that it is not easy to find sound positive definite sesquilinear forms satisfying Hermiticity (scalar products) given directly on functions depending on Grassmann variables, i.e. analogues to L^2 scalar products, although such examples exist [10]. This matter is the subject of the subsequent sections. In [11] a detour over the group (Hopf) algebra was proposed which we do not follow here.

5. Hilbert spaces of supersymmetric functions

In this section we present off-shell pre-Hilbert spaces and on-shell Hilbert spaces of functions in supersymmetry which are related to the L^2 and Sobolev spaces of analysis. It means that we look for positive sesquilinear forms by integrating superfunctions in all variables including the Grassmann ones. Let X, Y be supersymmetric functions of type (2.1) with regular coefficients. The key sesquilinear form is

$$\begin{aligned}(X, Y) &= \int d^8 z_1 d^8 z_2 \bar{X}(z_1) K(z_1, z_2) Y(z_2) \\ &= \int d^8 z_1 d^8 z_2 \bar{X}(z_1) [(P_c + P_a - P_T) K_0(z_1 - z_2)] Y(z_2) \\ &= \int d^8 z_1 d^8 z_2 \bar{X}(z_1) (P_c + P_a - P_T) K_0(z_1 - z_2) Y(z_2)\end{aligned}\quad (5.1)$$

where $P_i, i = c, a, T$ are the chiral, antichiral and transversal projections, respectively [1, 9] acting on the z_1 -variable (see equations to follow) and

$$\begin{aligned}K_0(z) &= \delta^2(\theta) \delta^2(\bar{\theta}) D^+(x) \\ D^+(x) &= \int e^{ipx} d\rho(p)\end{aligned}\quad (5.2)$$

$$= \int e^{-ipx} d\rho(-p)\quad (5.3)$$

where

$$\delta^2(\theta) = \theta^2, \quad \delta^2(\bar{\theta}) = \bar{\theta}^2.$$

We have used the standard notations

$$D_\alpha = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}^l \bar{\theta}^{\dot{\alpha}} \partial_l \quad (5.4)$$

$$D^\alpha = \epsilon^{\alpha\beta} D_\beta = -\partial^\alpha + i\sigma_{\alpha\dot{\alpha}}^l \bar{\theta}^{\dot{\alpha}} \partial_l \quad (5.5)$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^l \partial_l \quad (5.6)$$

$$\bar{D}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\beta}} = \bar{\partial}^{\dot{\alpha}} - i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^l \partial_l \quad (5.7)$$

for the covariant (and invariant) derivatives and

$$D^2 = D^\alpha D_\alpha = -(\partial^\alpha \partial_\alpha - 2i\partial_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial^\alpha + \bar{\theta}^2 \square) \quad (5.8)$$

$$\bar{D}^2 = \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} = -(\bar{\partial}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} + 2i\theta^\alpha \partial_{\alpha\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} + \theta^2 \square) \quad (5.9)$$

$$c = \bar{D}^2 D^2, \quad a = D^2 \bar{D}^2, \quad T = D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}} = -8\square + \frac{1}{2}(c + a) \quad (5.10)$$

$$P_c = \frac{1}{16\Box}c, \quad P_a = \frac{1}{16\Box}a, \quad P_T = -\frac{1}{8\Box}T. \tag{5.11}$$

The positive measure $d\rho(p)$ is concentrated in the interior of the forward light cone ($d\rho(-p)$ is concentrated in the interior of the backward light cone). It can be proved by direct computation [3] (see also remark (viii)) later in this section and section 7 of this paper) that (5.1) is positive definite. It is not strictly positive definite because it may have plenty of zero vectors. The well-known example is $d\rho(p) = \theta(p_0)\delta(p^2 + m^2)$, $d\rho(-p) = \theta(-p_0)\delta(p^2 + m^2)$ where m is the mass (the ‘massive’ case). In the massive case the zero vectors are harmless because they can be easily eliminated by the on-shell condition; the ‘massless’ case $m = 0$ requires a certain amount of extra work [3] but it is not too different from the massive one. We can also write

$$\begin{aligned} \langle X, Y \rangle &= \int d^8z_1 d^8z_2 [(P_c + P_a - P_T)K_0(z_1 - z_2)]\bar{X}(z_1)Y(z_2) \\ &= \int d^8z_1 d^8z_2 (P_c + P_a - P_T)K_0(z_1 - z_2)\bar{X}(z_1)Y(z_2) \end{aligned} \tag{5.12}$$

because $(P_c + P_a - P_T)K_0$ is even in the Grassman variables.

Beside being positive (for $X = Y$) the form (\cdot, \cdot) satisfies the usual (Hermiticity) property

$$\langle X, Y \rangle = \overline{\langle Y, X \rangle} \tag{5.13}$$

where the bar on the rhs means complex conjugation. Moreover we have

$$\overline{\langle Y, X \rangle} = \langle \bar{Y}, \bar{X} \rangle \tag{5.14}$$

where the bars on the rhs in (5.14) mean complex conjugation supplemented by Grassmann conjugation. The proofs including the positivity (i.e. the non-negativity of (\cdot, \cdot)) are by computation.

Some remarks are in order:

- (i) X, Y are functions (not fields). Their spinorial components are assumed to commute between themselves and with the Grassmann variables. Consequently we apply the mixed van de Waerden calculus which was worked out in [3] instead of the usual one.
- (ii) There is a surprising minus sign in front of P_T in (5.1). Reversing it produces an invariant, indefinite (not only semidefinite) sesquilinear form

$$\langle X, Y \rangle = \int d^8z_1 d^8z_2 \bar{X}(z_1)K_0(z_1 - z_2)Y(z_2) \tag{5.15}$$

because $P_c + P_a + P_T = 1$. This is the indication of an intrinsic Krein structure of the $N = 1$ superspace. The associated Hilbert space will be introduced below.

- (iii) The operators P_c, P_a, P_T in (5.1) can be moved to act on the z_2 -variable.
- (iv) The integral kernel in (5.15) depends only on $z_1 - z_2$ whereas the full integral kernel in (5.1) does not share this property.
- (v) Our positive sesquilinear form (5.1) is different from other proposals in supersymmetry (see [7] as well as [12] and the references given there) known to the author which seem to have the peculiar property of generating complex-valued norms.
- (vi) It is interesting to remark that the results of the relevant computation (like for instance the proof of positivity of (5.1) and Hermiticity (5.13)) is the same for X, Y being functions or fields. Whereas the results for functions are rigorous, those for fields, although computationally correct, must be looked at as formal.
- (vii) The operation of complex and Grassmann conjugation taken together is a conjugation operator in the Hilbert space.

(viii) An equivalent expression for the scalar product (\cdot, \cdot) is

$$\begin{aligned} (X, Y) &= \int d^8 z_1 d^8 z_2 [(P_c + P_a - P_T)\bar{X}(z_1)]Y(z_2) \\ &= \int d^4 x_1 d^4 x_2 D^+(x_1 - x_2) \\ &\quad \times \int d^2 \theta d^2 \bar{\theta} \bar{X}(x_1, \theta, \bar{\theta})[(P_c + P_a - P_T)Y](x_2, \theta, \bar{\theta}) \\ &= \int d^4 x_1 d^4 x_2 D^+(x_1 - x_2)(I_c + I_a - I_T) \end{aligned} \quad (5.16)$$

where the notations are self-explanatory. This form is convenient for computation, in particular for those needed in section 7.

(ix) Not only the supersymmetric invariant kernel induced by $P_c + P_a - P_T$ produces a positive result but also separately P_c , P_a and $-P_T$. Moreover positivity is also induced by $P_c + P_a + P_+ + P_-$ where $P_+ = \frac{D^2}{4\Box^{\frac{1}{2}}}$, $P_- = \frac{\bar{D}^2}{4\Box^{\frac{1}{2}}}$. The last property is connected to the positivity of the two-point function for the free Wess–Zumino model [3] in which the transversal sector is neglected.

Finally we factorize the zero-vectors of the positive sesquilinear form (5.1) and by completion obtain our Hilbert space. This factorization is for the case $d\rho(p) = \theta(p_0)\delta(p^2 + m^2)dp$, $m > 0$ harmless being equivalent to the on-shell restriction $p^2 + m^2 = 0$. This is not the case for $m = 0$ [3]. Altogether we generated an inherent Hilbert–Krein structure on $N = 1$ supersymmetric functions which we call the standard Hilbert–Krein structure of the superspace in order to distinguish it from other ones. The simple point of the whole business (which intuitively supports this paper) seems to be the minus sign in front of the transversal projection! The inherited bona fide Hilbert space is the analogue of the (on-shell) invariant L^2 -space. It is not a non-Hilbert generalization of a Hilbert space sometimes used in supersymmetries in which vectors can have complex lengths [12]. It is likely that it can be used in applications as for instance the Wigner-type representation theory on supersymmetric functions, supersymmetric canonical quantization (for first steps see [13]) and renormalization including gauge theories.

6. Supersymmetric Hilbert space operators and their adjoints

Now we come to supersymmetric operators and especially to their adjoints. Usual examples of formal supersymmetric operators are multiplication and especially differential (covariant and invariant) operators of the form

$$D_\beta, \bar{D}_{\dot{\beta}}, D^2, \bar{D}^2, \quad c = \bar{D}^2 D^2, \quad a = D^2 \bar{D}^2, \quad T = D^\beta \bar{D}^2 D_\beta = \bar{D}_{\dot{\beta}} D^2 \bar{D}^{\dot{\beta}}$$

etc. as well as analogue operators (supersymmetric generators) $Q_\beta, \bar{Q}_{\dot{\beta}}$ etc. Before starting let us compute

$$\begin{aligned} D_\beta X &= \varphi_\beta + \theta^\alpha (2m\epsilon_{\beta\alpha}) + \bar{\theta}_{\dot{\alpha}} (-v_\beta^{\dot{\alpha}} - i\sigma_\beta^{l\dot{\alpha}} \partial_l f) + \bar{\theta}^2 \left(\psi_\beta - \frac{i}{2} \sigma_{\beta\dot{\beta}}^l \partial_l \bar{\chi}^{\dot{\beta}} \right) \\ &\quad + \theta^\alpha \bar{\theta}^{\dot{\alpha}} (2\epsilon_{\alpha\dot{\beta}} \bar{\lambda}_{\dot{\alpha}} - i\sigma_{\beta\dot{\alpha}}^l \partial_l \varphi_\alpha) + \theta^2 \bar{\theta}_{\dot{\alpha}} (-i\sigma_\beta^{l\dot{\alpha}} \partial_l m) \\ &\quad + \bar{\theta}^2 \theta^\alpha \left(2\epsilon_{\beta\alpha} d + \frac{i}{2} \sigma_\beta^{l\dot{\beta}} \partial_l v_{\alpha\dot{\beta}} \right) - \frac{i}{2} \theta^2 \bar{\theta}^2 \sigma_{\beta\dot{\beta}}^l \partial_l \bar{\lambda}^{\dot{\beta}} \end{aligned} \quad (6.1)$$

and a similar expression for $\bar{D}_{\dot{\beta}} X$.

Note first that $D_\beta, \bar{D}_{\dot{\beta}}$ are true operators applied to functions X but no definite mathematical objects when applied to fields because for instance $2m\epsilon_{\alpha\beta}$ are no anticommuting

spinor components (for fixed β). A similar remark applies to the Q -operators. This does not affect us because we work in the frame of test functions in which the ‘fermionic’ components commute. We come to supersymmetric adjoint operators. Generally the Hilbert space adjoint A^+ of an operator A is (here and further on in this paper we disregard domain problems because as usual in elementary quantum field theory they turn out to be harmless) defined through

$$(X, AY) = (A^+X, Y). \quad (6.2)$$

This will be our definition too. It uses the supersymmetric true scalar product $(., .)$. As in [14], using the real bilinear form associated with (5.1), the transpose and the conjugate of an operator can be introduced but we do not need them.

We pass now to the most common examples. In order to study them we need some partial integration results [9]. We have

$$\int d^8z \bar{X} D_\beta Y = \mp \int d^8z (D_\beta \bar{X}) Y \quad (6.3)$$

$$\int d^8z \bar{X} \bar{D}_\beta Y = \mp \int d^8z (\bar{D}_\beta \bar{X}) Y \quad (6.4)$$

accordingly as \bar{X} (or X) is an even or odd Grassmann function. On the other hand we have from our previous considerations for functions

$$\overline{D_\beta X} = \mp \bar{D}_\beta \bar{X} \quad (6.5)$$

accordingly as \bar{X} (or X) is even or odd. Consequently

$$\int d^8z \bar{X} D_\beta Y = \int d^8z (\overline{D_\beta X}) Y \quad (6.6)$$

$$\int d^8z \bar{X} \bar{D}_\beta Y = \int d^8z (\overline{\bar{D}_\beta X}) Y \quad (6.7)$$

for arbitrary X, Y . Relations similar to (6.3)–(6.7) hold for the operators Q_β, \bar{Q}_β to be defined later in this section. Now we insert the kernels in the integrals. Indicating by superscripts the variables on which the operators act we have

$$D^{2(1)}(P_c + P_a - P_T)^{(1)} K_0(z_1 - z_2) = D^{2(2)}(P_c + P_a - P_T)^{(1)} K_0(z_1 - z_2) \quad (6.8)$$

$$\bar{D}^{2(1)}(P_c + P_a - P_T)^{(1)} K_0(z_1 - z_2) = \bar{D}^{2(2)}(P_c + P_a - P_T)^{(1)} K_0(z_1 - z_2). \quad (6.9)$$

Using (6.3)–(6.9) we get

$$(X, D^2 Y) = (\bar{D}^2 X, Y) \quad (6.10)$$

$$(X, \bar{D}^2 Y) = (D^2 X, Y) \quad (6.11)$$

i.e. \bar{D}^2 is the operator adjoint of D^2 and vice versa:

$$(D^2)^+ = \bar{D}^2, \quad (\bar{D}^2)^+ = D^2. \quad (6.12)$$

On the same lines we obtain as bona fide Hilbert space operators

$$c^+ = c, \quad a^+ = a, \quad T^+ = T \quad (6.13)$$

$$P_c^+ = P_c, \quad P_a^+ = P_a, \quad P_T^+ = P_T \quad (6.14)$$

as well as adjoint relations for the supersymmetric operator generators $Q_\beta^+ = \bar{Q}_\beta$, $(\bar{Q}_\beta)^+ = Q_\beta$ etc. The operators Q_α , \bar{Q}_α are given by

$$Q_\alpha = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}^l \bar{\theta}^{\dot{\alpha}} \partial_l \quad (6.15)$$

$$\bar{Q}_\alpha = -\bar{\partial}_\alpha - i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^l \partial_l. \quad (6.16)$$

Explicitly we have

$$\begin{aligned} Q_\beta X = & \varphi_\beta + \theta^\alpha (2m\epsilon_{\beta\alpha}) + \bar{\theta}_{\dot{\alpha}} (-v_\beta^{\dot{\alpha}} + i\sigma_\beta^{l\dot{\alpha}} \partial_l f) + \bar{\theta}^2 \left(\psi_\beta + \frac{i}{2} \sigma_{\beta\dot{\beta}}^l \partial_l \bar{\chi}^{\dot{\beta}} \right) \\ & + \theta^\alpha \bar{\theta}^{\dot{\alpha}} (2\epsilon_{\alpha\beta} \bar{\lambda}_{\dot{\alpha}} + i\sigma_{\beta\dot{\alpha}}^l \partial_l \varphi_\alpha) + \theta^2 \bar{\theta}_{\dot{\alpha}} (i\sigma_\beta^{l\dot{\alpha}} \partial_l m) \\ & + \bar{\theta}^2 \theta^\alpha \left(2\epsilon_{\beta\alpha} d - \frac{i}{2} \sigma_\beta^{l\dot{\beta}} \partial_l v_{\alpha\dot{\beta}} \right) + \frac{i}{2} \theta^2 \bar{\theta}^2 \sigma_{\beta\dot{\beta}}^l \partial_l \bar{\lambda}^{\dot{\beta}} \end{aligned} \quad (6.17)$$

and a similar relation for \bar{Q}_β .

Introducing the θ , $\bar{\theta}$ -expansion for X we get the action of Q , \bar{Q} on components which is well known in elementary supersymmetry.

In order to prove the operator adjoint relations $Q_\beta^+ = \bar{Q}_\beta$, $(\bar{Q}_\beta)^+ = Q_\beta$ we use the commutativity of Q and D operators and

$$Q_\beta^{(1)} (P_c + P_a - P_T)^{(1)} K_0(z_1 - z_2) = -Q_\beta^{(2)} (P_c + P_a - P_T)^{(1)} K_0(z_1 - z_2) \quad (6.18)$$

$$\bar{Q}_\beta^{(1)} (P_c + P_a - P_T)^{(1)} K_0(z_1 - z_2) = -\bar{Q}_\beta^{(2)} (P_c + P_a - P_T)^{(1)} K_0(z_1 - z_2). \quad (6.19)$$

Relations of type (6.18), (6.19) do not hold for the D_β , \bar{D}_β operators. This is the reason we cannot simply relate the D_β to the \bar{D}_β operators in our Hilbert space understanding. The same remark applies for multiplication and differentiation operators with respect to the Grassmann variables when considered separately. There is no undesirable consequence of this fact.

If P is the momentum operator then P , $\xi Q + \bar{\xi} \bar{Q}$, $aP + \xi Q + \bar{\xi} \bar{Q}$ are self-adjoint operators. Consequently the operator $\exp i(aP + \xi Q + \bar{\xi} \bar{Q})$ is a unitary operator ($U^+ = U^{-1}$, $(UX, UY) = (X, Y)$). Here we use the fact that Q commutes with P and D , \bar{D} . If $\tau = \tau(a, \xi, \bar{\xi})$ is the supersymmetric transformation in superspace then we can implement it unitarily on supersymmetric functions by the usual formula

$$U(\tau)X(z) = X(\tau(z)). \quad (6.20)$$

This is the rigorous version of a well-known formal statement ([15], p.91; see also [16]). If a Lorentz spin is present this formula changes as in [15]. Generally it is possible to construct the (irreducible) unitary representations of supersymmetry on functions (not on fields in which case the unitarity is formal [15, 16]) as this originally appears in the Wigner–Mackey theory for the Poincaré group. Measure-theoretic considerations in the Mackey theory of induced representations should be replaced by analogue considerations in the framework of von Neumann algebras.

Before ending this section let us point out that the supersymmetric framework can provide sometimes surprising results. Indeed being unitary the operator $\exp i(\xi Q + \bar{\xi} \bar{Q})$ studied above is bounded. But expanding it in powers of ξ , $\bar{\xi}$ it produces a sum of a finite number of powers of Q , \bar{Q} which at the first glance seems to be unbounded. But there is no contradiction because the boundedness is a consequence of the supersymmetric framework. This was already noted in [11].

7. Hilbert space realization on (multiplet) components

The supersymmetric fields appear in physics as multiplets of ordinary quantum fields, bosons and fermions. Whereas supersymmetry imposes some conditions on the ordinary fields in the multiplet (for instance the same number of bosons as fermions), the idea (at least in the perturbative framework) is to start with free multiplets on which an interaction (in the form of an interaction Lagrangian) is superimposed. The treatment makes use of supersymmetric technique or equivalently, if such a technique does not exist or is insufficiently developed, the work goes directly on component fields. At the first sight the Hilbert space of the free theory appears as a tensor product of the multiplet components. But this picture is too simple in order to be true. Indeed supersymmetry induces relations between multiplet components and a simple tensor product picture has to be abandoned. On the other hand being in the free case we expect nevertheless a Fock-type (tensor algebra) structure. How does it look like? This is the question which we do answer in this section in an even more general setting. We start with the standard Hilbert space of the $N = 1$ supersymmetry introduced in section 5 and decompose it on components. This is a matter of computation and we will give here only the results and some hints. The easiest way is to use (5.16) where we compute

$$I_c + I_a - I_T = \frac{1}{2} \bar{\varphi}_1 i \bar{\sigma}^l \partial_l \varphi_2 + \frac{1}{2} \chi_1 i \sigma^l \partial_l \bar{\chi}_2 + \frac{2}{\square} \lambda_1 i \sigma^l \partial_l \bar{\lambda}_2 + \frac{2}{\square} \psi_1 i \bar{\sigma}^l \partial_l \psi_2 + \frac{\square}{4} \bar{f}_1 f_2 + \bar{m}_1 m_2 + \bar{n}_1 n_2 + \frac{4}{\square} \bar{d}_1 d_2 - \bar{v}_1^l \partial_l \partial_k v_2^k + \frac{\square}{2} \bar{v}_1 v^{2l} \tag{7.1}$$

where $f_1(x_1) = f_1, f_2(x_2) = f_2, \varphi_1(x_1) = \varphi_1, \varphi_2(x_2) = \varphi_2$ etc. In order to see explicitly the positivity of the scalar product (5.12) we will take at a latter stage $X = Y$ which means $f_1 = f_2, \varphi_1 = \varphi_2$ etc. The contributions involving f, d, m, n in (7.1) are obviously positive (i.e. non-negative). The same holds for the v -contribution after partial integration. We have only to look at the φ, χ, λ and ψ -contributions. Let

$$J_1 = \int d^4 x_1 d^4 x_2 D^+(x_1 - x_2) \bar{\varphi}_1 (i \bar{\sigma}^l \partial_l) \varphi_2 \tag{7.2}$$

$$J_2 = \int d^4 x_1 d^4 x_2 D^+(x_1 - x_2) \chi_1 (i \sigma^l \partial_l) \bar{\chi}_2. \tag{7.3}$$

In order to study J_1, J_2 we go to the Fourier transform defined as

$$\tilde{f}(p) = \frac{1}{(2\pi)^2} \int e^{-ipx} f(x) d^4x$$

obtaining

$$\begin{aligned} J_1 &= \int d^4 p \tilde{\varphi}_1(p) \tilde{D}^+(p) i \bar{\sigma}^l \partial_l \tilde{\varphi}_2(-p) \\ &= \int d^4 p \tilde{\varphi}_1(-p) \rho(-p) i \bar{\sigma}^l (i p_l) \tilde{\varphi}_2(-p) \\ &= \int d^4 p \tilde{\varphi}_1(p) \rho(p) (\bar{\sigma}^l p_l) \tilde{\varphi}_2(p) \end{aligned} \tag{7.4}$$

where a Fourier normalization factor was omitted and we have used $\tilde{f}(-p) = \tilde{f}(p)$ and $\int d^4 x_1 d^4 x_2 D^+(x_1 - x_2) F(x_1) G(x_2) = \int d^4 p \tilde{F}(p) \tilde{D}^+(p) \tilde{G}(-p)$ where F, G are arbitrary functions. But $\bar{\sigma}^l p_l$ is positive in the forward light cone where $d\rho(p)$ is concentrated such that (for $\varphi_1 = \varphi_2$) J_1 is positive. In the same way J_2 is positive for $\chi_1 = \chi_2$ because $\sigma^l p_l$ is positive in the forward light cone too. Similar arguments hold for the term in λ, ψ in (7.1).

We conclude that (5.16) is for $X = Y$ positive (non-negative). The Hermiticity relations (5.13), (5.14) follow easily from the computation above keeping different X, Y and taking into account that $\sigma, \bar{\sigma}$ are self-adjoint.

Adopting appropriate normalizations we obtain the direct sum

$$H = \oplus H_{\text{components}} \quad (7.5)$$

where H is the Hilbert space of the $N = 1$ superspace and $H_{\text{components}}$ is the Hilbert spaces of the components (of bosonic and fermionic nature). The fermionic components (Weyl or Majorana) do not involve the mass term of their corresponding two-point function. The formula (7.5) simplifies for chiral, antichiral and transversal sectors. This seems to be an interesting result because it says that the standard supersymmetric Hilbert space is (because of supersymmetric invariance) a direct sum over the components. For a specific model like the vector field or the free Wess–Zumino model the elements in H are the ‘one-particle’ states and the Fock space is constructed as usual having the structure of a tensor algebra. This is similar to the case encountered in electrodynamics where the wavefunction of the (physical) transverse photon is the sum of two non-negative contributions but the Fock space continues to have the structure of a tensor algebra. Acting in H the supersymmetric operators, as expected, mix up the components. An example are the generators Q, \bar{Q} . Working out their action in (7.5) we re-obtain the usual supersymmetric transformations on multiplet components.

8. The simplest example: free supersymmetric quantum field

The first model of a supersymmetric quantum field theory in four dimensions was the chiral–antichiral Wess–Zumino model with a third power interaction. Without the interaction the Wess–Zumino (free) field is still a chiral–antichiral one. We prefer instead of the Wess–Zumino model the free massive vector field $V(z)$ which we define here by the simplest supersymmetric invariant two-point function which at the same time is positive definite. The corresponding kernel in the sense of (5.1) is induced by $P_c + P_a - P_T$ (the generalization $\zeta_c P_c + \zeta_a P_a - \zeta_T P_T$ with $\zeta_i \geq 0$ is also possible). We start by defining representations of supersymmetries on supersymmetric functions by

$$U(\tau)X(z) = X(\tau(z)) \quad (8.1)$$

where $\tau = \tau(z)$ is a supersymmetric transformation. Considered in the Krein space, i.e. in the space with inner product $\langle \cdot, \cdot \rangle$ (see (5.15)), as this is usually suggested in the literature on the subject [15, 16], this representation is not unitary. We obtain a unitary representation when representing in the corresponding Hilbert space H with scalar product (\cdot, \cdot) (see (5.1)). The supersymmetric quantum field $V(z)$ is similar to the scalar neutral field in the common quantum field theory. Its definition requires a (supersymmetric) symmetric Fock space. Indeed we construct the symmetric Fock space on H which we denote by $\mathcal{F} = \text{Fock}(H)$ (antisymmetric Fock spaces are reserved for ghosts). A general element of \mathcal{F} will be denoted by $\Phi = (\Phi^{(0)}, \Phi^{(1)}, \dots, \Phi^{(n)}, \dots)$, $\Phi^{(0)} = 1$, $\Phi^{(n)} = \Phi^{(n)}(z_1, z_2, \dots, z_n)$. Note that working with the overall anticommuting convention discussed above, Fock spaces in supersymmetry in the scalar case (complying with the right statistics) are always symmetric. The above unitary representation can be extended as usually to the Fock space \mathcal{F} . We set for the vector field V smeared with the (test) supersymmetric function $X(z)$:

$$V(X) = V^+(X) + V^-(X) \quad (8.2)$$

with

$$(V^+(X)\Phi)^{(n)}(w_1, \dots, w_n) = \sqrt{n+1}X(w), \Phi^{(n+1)}(w, w_1, \dots, w_n) \quad (8.3)$$

$$(V^-(X)\Phi)^{(n)}(w_1, \dots, w_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^n X(w_j)\Phi^{(n-1)}(w_1, \dots, \hat{w}_j, \dots, w_n) \tag{8.4}$$

where $w = (p, \theta, \bar{\theta})$ and p is the momentum. The adjoint of V is given by $V^+(X) = V(\bar{X})$, i.e. we have a real (neutral) supersymmetric field. We can project on components using the representation (7.5) of the Hilbert space H but we are not interested in this question. The n -point functions can be given as usual as products of two-point functions. They are non-vanishing only for even n and are even functions of the Grassmann variables in the combination $\theta\bar{\theta}$. This follows from the relation (see for instance [13])

$$\begin{aligned} (P_c + P_a - P_T)\delta^2(\theta_1 - \theta_2)\delta^2(\bar{\theta}_1 - \bar{\theta}_2) &= (1 - 2P_T)\delta^2(\theta_1 - \theta_2)\delta^2(\bar{\theta}_1 - \bar{\theta}_2) \\ &= \frac{4}{\square}(1 - i\theta_1\sigma^l\bar{\theta}_2\partial_l - i\bar{\theta}_1\bar{\sigma}^l\theta_2\partial_l) \\ &\quad + \theta_1^2\bar{\theta}_2^2 + \bar{\theta}_1^2\theta_2^2 + 2(\theta_1\sigma^l\bar{\theta}_1) \left(\theta_2\sigma_l\bar{\theta}_2 + \frac{2}{\square}\partial_l\delta^m\bar{\theta}_2\bar{\sigma}_m\theta_2 \right) \\ &\quad - i\theta_1^2\bar{\theta}_1\bar{\sigma}^l\theta_2\bar{\theta}_2^2\partial_l - i\bar{\theta}_1^2\theta_1\sigma^l\bar{\theta}_2\theta_2^2\partial_l + \frac{1}{4}\square\theta_1^2\bar{\theta}_1^2\theta_2^2\bar{\theta}_2^2. \end{aligned} \tag{8.5}$$

This last property which in fact, because of supersymmetric invariance, holds in any supersymmetric quantum field theory (see the next section) makes obsolete the problems connected to nuclearity which have been discussed in section 3.

9. Implications of supersymmetric invariance to the n -point functions

In this section we give the general form of a supersymmetric invariant n -point function generalizing a result in [13]. Suppose we have a unitary representation $U(\tau)$ of the full supersymmetric group in a Hilbert space of supersymmetric functions

$$U(\tau)X(z) = X(\tau(z)). \tag{9.1}$$

A quantum supersymmetric field (free or interacting) and its vacuum are supposed to be supersymmetric invariant such that this property is shared by the vacuum expectation values too. Denoting by $W_n(z_1, z_2, \dots, z_n)$ the n -point function (considered as superdistribution) we have simultaneous relations

$$Q_i W_n(z_1, z_2, \dots, z_n) = 0, \quad \bar{Q}_i W_n(z_1, z_2, \dots, z_n) = 0 \tag{9.2}$$

for $i = 1, 2, \dots, n$ where Q_i, \bar{Q}_i are the supersymmetric generators on the variables z_i . Because of translation invariance W_n depend on $x_i, i = 1, 2, \dots, n$ through differences $x_i - x_{i+1}$. How does the dependence of W_n on $\theta_i, \bar{\theta}_i$ look like? We start with the case $n = 2$. Simultaneous supersymmetric invariance implies for $W_2(z_1, z_2)$

$$(Q_1 + Q_2)W_2(z_1, z_2) = 0 \tag{9.3}$$

$$(\bar{Q}_1 + \bar{Q}_2)W_2(z_1, z_2) = 0 \tag{9.4}$$

where Q_1, Q_2 and \bar{Q}_1, \bar{Q}_2 act on the variable z_1 and z_2 , respectively.

In order to solve these equations we introduce new variables $\theta = \frac{1}{2}(\theta_1 + \theta_2)$ and $\zeta = \theta_1 - \theta_2$ together with their conjugates as well as (by translation invariance) the difference variable $x = x_1 - x_2$ (for a similar argument see [17]). Note that by introducing the difference variable $x = x_1 - x_2$ the derivative ∂_l in Q_2, \bar{Q}_2 taken with respect to the second variable changes sign such that in the new variables equations (9.2) for $n = 2$ take the form

$$\left(\frac{\partial}{\partial \theta^\alpha} - i \sigma_{\alpha\dot{\alpha}}^l \bar{\zeta}^{\dot{\alpha}} \partial_l \right) W_2 = 0 \quad (9.5)$$

$$\left(\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \zeta^\alpha \sigma_{\alpha\dot{\alpha}}^l \partial_l \right) W_2 = 0 \quad (9.6)$$

where W_2 is a function of $x = x_1 - x_2, \theta, \bar{\theta}, \zeta, \bar{\zeta}$. We want to solve this system of partial differential equations in mixed commutative and non-commutative variables. A first (trivial and from the physical point of view uninteresting) solution for W_2 is a constant. Other solutions can be obtained in a two-step procedure by using in the first step equation (9.5) to factorize from W_2 the exponential $\exp(i\theta\sigma^l\bar{\zeta}\partial_l)$ (for more details see also [13]). We write

$$W_2 = \exp(i\theta\sigma^l\bar{\zeta}\partial_l) D.$$

The first equation (9.5) implies $\frac{\partial}{\partial \theta^\alpha} D = 0, \alpha = 1, 2$ which means that D is independent of $\theta_\alpha, \alpha = 1, 2$. In the second step we write

$$D = \exp(-i\zeta\sigma^l\bar{\theta}\partial_l) E$$

use (9.6), and conclude as above that E is independent not only of θ but also independent of $\bar{\theta}$. These two exponentials cover the dependence of W_2 from the variables θ and $\bar{\theta}$. The residual dependence in E is in $\zeta, \bar{\zeta}$ and $x = x_1 - x_2$. Altogether the general solution of equations (9.5), (9.6) in the $x, \theta, \bar{\theta}, \zeta, \bar{\zeta}$ -variables is (with the exception of the constant solution) of the form

$$W_2(x, \theta, \bar{\theta}, \zeta, \bar{\zeta}) = \exp[-i(\zeta\sigma^l\bar{\theta} - \theta\sigma^l\bar{\zeta})\partial_l] E(x, \zeta, \bar{\zeta}) \quad (9.7)$$

where from invariance considerations E is restricted to

$$E(x, \zeta, \bar{\zeta}) = E_1(x) + \zeta^2 E_2(x) + \bar{\zeta}^2 E_3(x) + \zeta\sigma^l\bar{\zeta}\partial_l E_4(x) + \zeta^2\bar{\zeta}^2 E_5(x) \quad (9.8)$$

with $E_i(x) = E_i(x_1 - x_2), i = 1, \dots, 5$ being Lorentz invariant functions (or distributions). The reality condition would require real $E_i, i = 1, \dots, 5$ as well as $E_2 = E_3$. Translating back to the $\theta, \bar{\theta}$ -variables it is possible to relate (9.7) to the known invariants constructed with the help of the five invariant operators $P_i, i = c, a, T, +, -$ [1].

In the next section we will use this result in order to discuss a supersymmetric Källén-Lehmann representation for a scalar neutral supersymmetric quantum field theory. We continue this section by extended our result from $n = 2$ to general n . This is easily done by introducing the new variables

$$\theta = \frac{1}{n}(\theta_1 + \dots + \theta_n), \quad \zeta_i = \theta_i - \theta_{i+1}, \quad i = 1, 2, \dots, n-1 \quad (9.9)$$

$$\bar{\theta} = \frac{1}{n}(\bar{\theta}_1 + \dots + \bar{\theta}_n), \quad \bar{\zeta}_i = \bar{\theta}_i - \bar{\theta}_{i+1}, \quad i = 1, 2, \dots, n-1 \quad (9.10)$$

such that equations (9.5), (9.6) transform to

$$\left(\frac{\partial}{\partial \theta^\alpha} - i \sum_{i=1}^{n-1} \sigma_{\alpha\dot{\alpha}}^{l_i} \bar{\zeta}_i^{\dot{\alpha}} \partial_{l_i} \right) W_n = 0 \quad (9.11)$$

$$\left(\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \sum_{i=1}^{n-1} \zeta_i^\alpha \sigma_{\alpha\dot{\alpha}}^{l_i} \partial_{l_i} \right) W_n = 0. \quad (9.12)$$

These equations can be solved in a two-step procedure exactly as in the previous case. The only difference is that in the exponentials in (9.11), (9.12) we have to consider the corresponding

sums over $\zeta_i, \bar{\zeta}_i, i = 1, \dots, n - 1$. The result is

$$W_n(z_1, z_2, \dots, z_n) = \exp \left[-i \sum_{i=1}^{n-1} (\zeta_i \sigma^l \bar{\theta} - \theta \sigma^l \bar{\zeta}_i) \partial_{l_i} \right] \\ \times E(x_1 - x_2, \dots, x_{n-1} - x_n, \zeta_1, \bar{\zeta}_1, \dots, \zeta_{n-1}, \bar{\zeta}_{n-1}). \quad (9.13)$$

The function E turns out as above, from invariance considerations, to be a finite sum of products of $\theta_i \theta_j, \bar{\theta}_i \bar{\theta}_j, \theta_i \sigma^l \partial_l \bar{\theta}_j$ multiplied (applied) to Lorentz invariant functions (distributions) depending on difference variables $x_i - x_j$. The reality condition imposes further obvious restrictions.

10. Some other aspects of quantum superfields

We come to the point of putting together the ingredients and experience developed in the preceding sections in order to indicate some other aspects of supersymmetric quantum fields which are accessible to our rigorous methods. This can be done following ideas in the axiomatic quantum field theory [4]. Quantum fields are defined as operator-valued superdistributions. The supersymmetric invariance is formulated with the help of an unitary representation of the super-Poincaré group on the postulated Hilbert space of supersymmetric functions. Other axioms [4], including positivity and locality (or weak locality), make no problems. As in the usual case the reconstruction theorem based on the *-algebra of section 4 enables us to realize the supersymmetric quantum fields as operator-valued distributions on supersymmetric test function spaces starting from the n -point functions. Before continuing on this line let us remark that our generalization is not trivial. It does not seem to confirm the impression that in order to mathematically extend a theory to the supersymmetric case we only have to adjust the prefix ‘super’ on the right places. Indeed, even the simple examples of the supersymmetric free field in section 8 as well as the related generalized free field later on in this section show that, among others, our construction is centred on a non-trivial bona fide Hilbert space of supersymmetric functions and more important, that this Hilbert space is intimately connected with an inherent Hilbert–Krein structure of the superspace [3].

We continue to restrict the consideration to the real massive vector field, which using another terminology, could be called the supersymmetric scalar neutral field. As a first application we can mention the supersymmetric Källén–Lehmann representation. Indeed the results of section 9 show that the most general supersymmetric invariant two-point function is given as in (5.1) with the help of a kernel constructed with a linear combination of the invariant projections $P_i, i = c, a, T, +, -$. An example is the standard kernel $P_c + P_a - P_T$ in (5.1) but it turns out that this is not the most general one compatible with positivity. By computation it can be proved that (in the massive case) the most general invariant and positive kernel is generated by $\lambda_c P_c + \lambda_a P_a + \lambda P_+ + \bar{\lambda} P_- - \lambda_T P_T$ with $\lambda_c, \lambda_a, \lambda_T$ being positive (or zero) and $|\lambda^2| < \lambda_c \lambda_a$. This result allows us to introduce by known formulae [18] in analogy to (8.2)–(8.4) a whole family of (supersymmetric) quantum fields called generalized free fields. We will not discuss them in detail here.

Let us remind the reader that from a historical point of view the two main achievements of the axiomatic quantum field theory were the PCT and the spin and statistics theorems [4, 18]. We end this paper by formulating the PCT result in the supersymmetric framework. First besides the P, C, T transformations in quantum field theory (they do not act the Grassmann variables; they act only the multiplet components) we introduce the Θ

transformation by [14]

$$\begin{aligned}\theta_\alpha &\rightarrow i\bar{\theta}'_\alpha, & \bar{\theta}_\alpha &\rightarrow i\theta'_\alpha \\ \theta^\alpha &\rightarrow i\bar{\theta}'^\alpha, & \bar{\theta}^\alpha &\rightarrow i\theta'^\alpha.\end{aligned}\tag{10.1}$$

This is the *PCT* transformation of the spinor θ with components θ_α [4]. The supersymmetric *PCT* theorem means the invariance of the theory to the transformation *PCT* supplemented by Θ (it could be also called the Θ *PCT* theorem). The proof in our framework follows for example from the general representation of the n -point functions in the scalar neutral theory given in (9.13). Indeed by locality the spacetime coefficients of the Grassmann variable products in the n -point functions (9.13) satisfy *PCT* invariance. On the other hand, taking into account that $\theta_i\sigma^i\bar{\theta}_j\partial_l$ and $\theta_i\theta_j + \bar{\theta}_i\bar{\theta}_j$ are invariant under Θ it follows that the supersymmetric n -point functions altogether are Θ *PCT* invariant.

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